THE SPECIAL TRANS FUNCTIONS THEORY FOR THE DEGREE OF THE NUCLEAR FUEL BURN-UP ESTIMATION*

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Abstract. *The problem of finding an exact analytical closed form solution to the degree of the nuclear fuel burn-up simple transcendental equation is studied in some detail, by using the Special trans functions theory (STFT). Structure of the STFT solutions, derivations, numerical results and graphical simulations confirm the validity and base principle of the STFT.*

Key words: *The degree of the nuclear fuel burn-up, the special trans functions theory-STFT, analytical closed form solution, advanced STFT iterative procedure, special* function.

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1. INTRODUCTION

The subject of our interest presented within this article is oriented towards determining of the degree of the nuclear fuel burn-up, for some simple cases, by using the Special trans functions theory (STFT). Namely, the modified Lambert nonlinear functional equation, which describes the degree of the nuclear fuel burn-up, is solved analytically in closed form (as a special trans function, $\mathrm{tran}_{NF}(N,\alpha,\lambda_1,\lambda_2,t)$ or, $\mathrm{tran}_{NF}(D,d)$). Also, the Lambert equation is solved by using an advanced iterative procedure, within the STF theory. Let us note that, for instance, this theoretical problem is defined in [1 -5]. Among firstly published methods available in literature, attention was focused on measuring the neutron radiation. In references, [1, 2], in some detail, the problem is described, as follows: The degree of the nuclear fuel burn-up is necessary parameter for the analysis of the following problems in nuclear energetic:

- The optimal arrangement of spent fuel structures in laying down and keeping basins;
- Insuring of the nuclear safety and control while the nuclear fuel reprocessing,
- The Irradiation risk prognosis;
- Examining of the Irradiation behavior of the material;
- The fulfillment of the agreement of the nuclear weapons nonexttinslon, especially connected with the export of nuclear power plants;
- Making possible the maximal generating of energy.

The advantage of this way of control (the method based on the measurement of the neutron radiation) is reflected in high sensitivity, simplicity and reliability of the technical equipment, operability. In addition, the degree of the nuclear fuel burn-up is a parameter of vital importance in the description of the irradiation behavior of the material.

In contrast to the gamma-spectrometry method, the neutron measurements can be performed, practically, immediately after the drawing of fuel structures out of the reactor, as it is enough to enable only the necessary irradiation stability of the detector on γ radiation. In spent fuel structures, as a source of neutrons, long living nuclides prevail over what permits the burn-up control at any laying down time which represents the practical interest. This also contributes to the role of the history of radiation on the interpretation of results, to be decreased.

The dependence of the specific total neutron output on the burn-up degree in the diapason ω>15 MWd/kg is defined by the spontaneous fission of the curium nuclides $^{242}\mathrm{Cm}$ and $^{244}\mathrm{Cm}$ where the contribution $^{242}\mathrm{Cm}$ is irrelevantly small at the laying down time t>3 years due to the short half - life (0.45 years). The contribution of the (α, n) reaction is essential only at small burn-up degrees (ω<10Mwd/kg), and if, at the same time, the laying down time is great (t>10 years). It becomes predominant due to $^{241}\mathrm{Am}$ formation.

The dependence of the total neutron output on the burn-up degree is a very nonlinear function, what complicates the measurement of the mean burn-up degree according to the quantity of spent fuel structures in the neutron measurement order. Namely, the computer program processing of results of discrete or continual measurements of the neutron flux is necessary for the burn-up profile measurement. The processing algorithm is essentially simplified under the condition when spent fuel structures are laid down for more than three years (t>3 years), and when burn-up degree values are ω>15MWd/kg. In that case the

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counting speed of the detector is connected with the burn-up degree by the following relation

$$N = N_0 \left(\frac{\omega}{\omega_0} \right)^n \exp(-\lambda t)$$

where N_o and ω_o are standardized constants; λ is the decay constant ^{244}Cm; n = 4.1 – 4.3; t is the laying down time counted from the moment of the three years period of laying down. However, in a general case it is necessary to solve the transcendental equation of the type

$$N = \left(\frac{\omega}{3.266 \cdot \alpha^{0.445}} \right)^{n_1} \cdot e^{-\lambda_1 t} + \left(\frac{\omega}{0.272 \cdot \alpha^{0.445}} \right)^{n_2} \cdot e^{-\lambda_2 t} \quad (1)$$

where the first term presents the contribution ^{244}Cm 244 in the neutron radiation counting speed. The second term presents the contribution ^{242}Cm.

1.1. Concerning the degree of the nuclear fuel burn-up transcendental equation

The subject of the theoretical analysis presented within this subsection is analytical in closed form solving of the Eq.(1), where N is the neutron radiation counting speed, ω denotes the burn-up degree, α is the initial fuel enrichment in ^{235}U, and t is the laying down time after taking fuel elements and structures out of the reactor (i.e. after finishing of irradiation). Exponents n_1, n_2 are: $n_1 = 4.1$, $n_2 = 1.75$, while the decay constants are $\lambda_1 = 0.0382$ and $\lambda_2 = 1.55$, per year, respectively.

In addition, the empiric coefficients 3.266 and 0.272 take care of different specific neutron outputs for these nuclides as well as of the unit's proportion and standardization.

After structural modification Eq. (1) takes the form

$$A\omega^{n_1} + B\omega^{n_2} = 1 \quad (2)$$

where

$$A = \left(\frac{1}{N \cdot 3.266 \cdot \alpha^{0.445}} \right)^{n_1} \cdot \exp(-\lambda_1 t)$$

$$B = \left(\frac{1}{N \cdot 0.272 \cdot \alpha^{0.445}} \right)^{n_2} \cdot \exp(-\lambda_2 t).$$

Or, for given numeric parameters, we have

$$A = \left(\frac{1}{N \cdot 3.266 \cdot \alpha^{0.445}} \right)^{4.1} \cdot \exp(-0.0382 \cdot t)$$

$$B = \left(\frac{1}{N \cdot 0.272 \cdot \alpha^{0.445}} \right)^{1.75} \cdot \exp(-1.55 \cdot t).$$

Analogically, from Eq. (2) after simple modifications, we have

$$\psi + D\psi^d = 1 \quad (3)$$

where

$$\psi = B\omega^{n_2}, \quad d = \frac{n_1}{n_2} > 1, \quad D = \frac{A}{B^d}. \quad (4)$$

Consequently, from Eq. (4), formula for burn-up degree takes the form:

$$\omega = \left(\frac{\psi}{B} \right)^{\frac{1}{n_2}} \textit{ and } \quad \omega = \left(\frac{\psi}{B} \right)^{\frac{1}{1.75}}. \quad (5)$$

The modified Lambert nonlinear functional equation of transcendental type (Eq. (3)) is solvable using the Special Tran Function Theory [6-23]. Let us note that Perovich's Special Tran Functions Theory (STFT), has been proved to be a very powerful theory for solving transcendental equations, some integral and integro-differential equations, and, obtaining exact analytical closed-form solutions. Examples of STFT application are shown in articles concerning the closed-form solution genesis in: the theory of neutron slowing down [[6],[7],[16]], the nonlinear circuit theory [[8],[16],[22]], the linear transport theory [[9],[16],[20]], the Hopfield neuron analysis [11], some families of transcendental equations analysis [[12],[16],[17]], the solar cell theoretical analysis [[13],[18],[22]], the Plutonium temperature estimation [14], the ambient temperature estimation [[15], [16]], the Lambert transcendental equations analysis [[16],[17],[23]], as well as in the problem in the engineering materials [19], [21],[22]], etc.

Let us note that in more general cases Eq. (1) takes the form

$$N = \sum_{k=1}^{M} \left(\frac{\omega}{e_k \cdot \alpha_k^{0.445}} \right)^{n_k} \cdot \exp(-\lambda_k t)$$

where M is number of nuclides. Determining analytical closed form solution to the above transcendental equation will be the subject of our further research.

2. OBTAINING AN ANALYTICAL CLOSED FORM SOLUTION TO THE MODIFIED LAMBERT TRANSCENDENTAL EQUATION (3)

This section contains an analytical closed form solution to the Eq. (3), obtained by using the STFT. Namely, in several references [[16], [17], [23] et al], the modified Lambert transcendental Eq. (3) is solved by direct application of the STFT. Consequently, we have

$$\psi = tran_{NF}(D, d)$$

where $tran_{NF}(D,d)$ is a new special function defined as

$$\mathrm{tran_{NF}}(D,d)=\lim_{x\to\infty}\left(\frac{D\Phi(x-1,D,d)+1}{D\Phi(x,D,d)+1}\right) \qquad (6)$$

where functional series Φ(x, D, d) takes the form

$$\Phi(x,D,d)= \qquad (7)$$

$$\sum_{n=0}^{\left[\frac{x}{d}\right]}(1+D)^n+\sum_{n=\left[\frac{x}{d}\right]+1}^{[x]}\sum_{k=0}^{[(x-n)(d-1)]}\frac{D^k n!}{(n-k)!k!}$$

where [x] denotes the greatest integer less than or equal to x. Now, more explicitly, from equations (5), (6) and (7) we have

$$\psi=\lim_{x\to\infty}\left(\frac{D\sum_{n=0}^{\left[\frac{x-1}{d}\right]}(1+D)^n+D\sum_{n=\left[\frac{x-1}{d}\right]+1}^{[x-1]}\sum_{k=0}^{[(x-1-n)(d-1)]}\frac{D^k n!}{(n-k)!k!}+1}{D\sum_{n=0}^{\left[\frac{x}{d}\right]}(1+D)^n+D\sum_{n=\left[\frac{x}{d}\right]+1}^{[x]}\sum_{k=0}^{[(x-n)(d-1)]}\frac{D^k n!}{(n-k)!k!}+1}\right). \qquad (8)$$

For practical analysis and numerical calculations by Mathematica program application the formula (6) takes the form

$$\langle\psi\rangle_P=\left\langle\frac{D\Phi(x-1,D,d)+1}{D\Phi(x,D,d)+1}\right\rangle_P \qquad (9)$$

where $\langle\psi\rangle_P$ denotes the numerical value of the transcendental number ψ given with P accurate digits, where P is defined as

$$P=[1-\log(\mathrm{abs}(G))] \qquad (10)$$

and, Error function G is defined as

$$G=\psi+D\psi^d-1.$$

More explicitly the formula (9) takes the form

$$\langle\psi\rangle_P=\left\langle\frac{D\sum_{n=0}^{\left[\frac{x-1}{d}\right]}(1+D)^n+D\sum_{n=\left[\frac{x-1}{d}\right]+1}^{[x-1]}\sum_{k=0}^{[(x-1-n)(d-1)]}\frac{D^k n!}{(n-k)!k!}+1}{D\sum_{n=0}^{\left[\frac{x}{d}\right]}(1+D)^n+D\sum_{n=\left[\frac{x}{d}\right]+1}^{[x]}\sum_{k=0}^{[(x-n)(d-1)]}\frac{D^k n!}{(n-k)!k!}+1}\right\rangle_P \qquad (11)$$

Note that computational complexity for determining numerical value $\langle\psi\rangle_P$, from equation (11), depends of nuclides parameters D and d. For some real values of parameters D and d, computational complexity is not easy. Note that some numerical results for W (formally - mathematical identical physical phenomenon forψ), based on equation (11), are given in Table I in [23]. It is more than clear that STFT works! Unfortunately, for some real parameters D and d, the sumlimit in the formulae (11), within Mathematica program, is not small. Consequently, an advanced STFT iterative procedure is presented in next section.

Note, that sections 2 and 3, in this article, are identical with sections II and III in [23], in formally-mathematical sense. Let us note that between physical phenomenon of the nuclear fuel burn-up and physical phenomenon of current in the RC diode circuit there is absolutely no correlation.

3. STFT ADVANCED ITERATIVE PROCEDURE FOR SOLVING TRANSCENDENTAL EQUATION (3)

The subject of the theoretical analysis presented here is obtaining a solution to the modified Lambert transcendental equation (3) with arbitrary number of accurate digits in the numerical structure, by using the advanced STFT iterative procedure.

The outline of the iteration process begins with the certain value ofψ. After that, the second value of ψ is obtained from Eq. (3) in the form ${}^{(2)}\psi=1-D\left({}^{(1)}\psi\right)^d$ If ${}^{(2)}\psi$ does not satisfy the error criterion $\left|{}^{(2)}\psi-{}^{(1)}\psi\right|<\varepsilon$, where ε is an arbitrary small real number, then a new value of ψ is found from equation (3) using ${}^{(2)}\psi$ and the whole procedure is repeated. Let us note that general scheme of the advanced STFT iterative procedure takes the form

$${}^{(n)}\psi=1-D\left({}^{(n-1)}\psi\right)^d. \qquad (12)$$

where n is number of iteration. For instance, if number of iteration is 10 then the advanced STFT iterative formula takes the form

$${}^{(10)}\psi\approx$$
$$1-D(1-D(1-D\ldots((\ldots.(1-D(1-DD_2^d)^d)^d)^d)^d)^d)^d. \qquad (13)$$

Or, for N iteration we have

$${}^{(N)}\psi\approx$$
$$1-D(1-D(1-D(1-D(1-D(1-D(1-D(1-D(\ldots\ldots$$
$$1-D\ldots_{(N-4)}\ldots(1-DD_2^d)^d)^d)^d)^d)^d)^d)^d)^d\ldots_{(N-4)}$$

where

$$D_2 = 1 - D\left(1 - D^1\psi^d\right)^d$$

Let us note that from Eq. (5) value of the nuclear burn-up degree is estimated with arbitrary number of accurate digits in the numerical structure. Namely, expression to the degree of nuclear fuel burn-up takes the form

$$\omega = \left(\frac{1}{B}\right)^{\frac{1}{n_2}} \cdot \psi^{\frac{1}{n_2}} = \beta \cdot \psi^{\frac{1}{n_2}}, \quad \beta = \left(\frac{1}{B}\right)^{\frac{1}{n_2}}$$

or, more explicitly

$$\omega = \beta \begin{pmatrix} A(1-D(1-D(1-D(1-D(1-D(1-D(\ldots. \\ 1-D\ldots_{(N-4)puta}\ldots(1-DD_2^d)^d)^d)^d)^d)^d\ldots_{(N-4)puta} \end{pmatrix}^{\left(\frac{1}{n_2}\right)} . \qquad (14)$$

4. NUMERICAL RESULTS

In this section, for the practical numerical analysis of Eq. (14), we have used the following nuclides parameters: $n_1 = 4.1$, $n_2 = 1.75$, while the decay constants are $\lambda_1 = 0.0382$ and $\lambda_2 = 1.55$, per year, respectively.

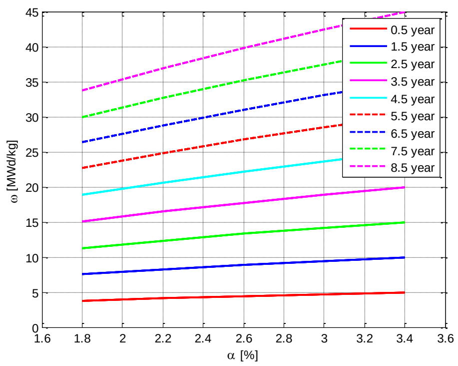

Figure 1. The degree of nuclear fuel burn-up as a function of the initial fuel enrichment in ^{235}U, for various values of t

Let us note that obtained numerical results obtained by using the Mathematica program and its graphical results are given in Table 1, and in Figs. 1,2 and 3.

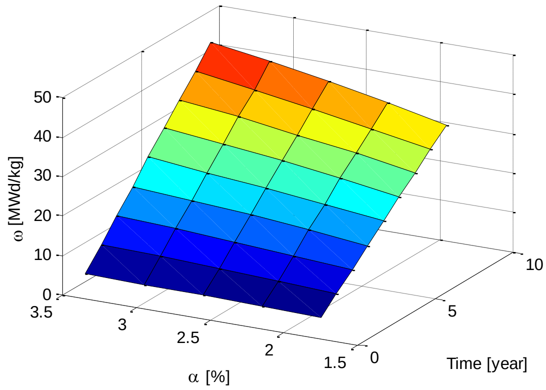

Figure 2. Graphical 3D presentation of the function $\omega = f(\alpha, t)$

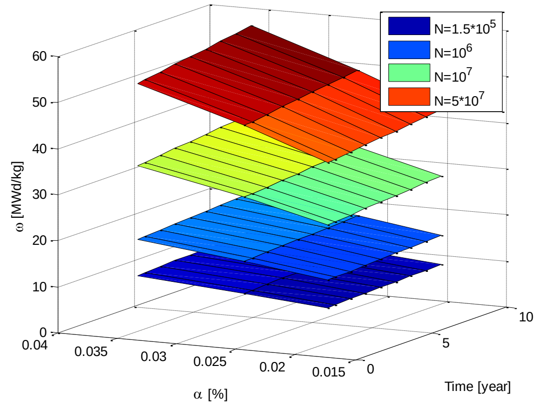

Figure 3. Graphical 3D presentation of the function $\omega = f(\alpha, t)$ for various values of N

Table 1. Numerical values to the degree of the nuclear fuel burn-up and its precision P, obtained for various values of t, α and N

t year	N	α	ω	P
0.5	3700	0.018	3.757569...	16
		0.022	4.108552...	17
		0.026	4.425617...	17
		0.030	4.716607...	16
		0.034	4.986764...	16
1.5	45000	0.018	7.530785...	18
		0.022	8.234213...	16
		0.026	8.869663...	17
		0.030	9.452854...	17
		0.034	9.994294...	16
2.5	225000	0.018	11.29761...	16
		0.022	12.35289...	16
		0.026	13.30619...	17
		0.030	14.18109...	17
		0.034	14.99335...	16
3.5	700000	0.018	15.04496...	17
		0.022	16.45027...	18
		0.026	17.71977...	16
		0.030	18.88486...	16
		0.034	19.96655...	16

4.5	$1.7\cdot10^6$	0.018	18.85560...	16
		0.022	20.61685...	18
		0.026	22.20789...	17
		0.030	23.66808...	17
		0.034	25.02374...	17
5.5	$3.5\cdot10^6$	0.018	22.69766...	18
		0.022	24.81778...	16
		0.026	26.73302...	18
		0.030	28.49075...	16
		0.034	30.12264...	17
6.5	$6.2\cdot10^6$	0.018	26.33871...	17
		0.022	28.79893...	17
		0.026	31.02140...	16
		0.030	33.06109...	16
		0.034	34.95476...	16
7.5	$1.0\cdot10^7$	0.018	29.87288...	17
		0.022	32.66321...	16
		0.026	35.18390...	16
		0.030	37.49728...	17
		0.034	39.64505...	17
8.5	$1.6\cdot10^7$	0.018	33.81496...	16
		0.022	36.97351...	16
		0.026	39.82683163…	17
		0.030	42.44549245…	16
		0.034	44.87668168…	17

5. CONCLUSIONS

From the previous sections it is obvious that the Special Tran Functions Theory is a consistent approach to solving transcendental equations in the degree of nuclear fuel burn-up domain for defined case. This means that we can obtain a new special function $\mathrm{tran}_{NF}(D,d)$. New formulae within nuclear fuel burn-up theory, Eqs. (11) and (14) being derived in the paper, using the STFT, is valid in the numerical sense (See Table 1). Thus, obtained analytical solutions apart from theoretical value have practical application.

The theoretical accuracy of the STFT ([6-23] is unlimited, and, in physical sense optimal precision is attainable with this approach (See Table 1).

Also, a new, original STFT advanced iterative procedure for determination of the degree of the nuclear fuel burn-up with optimal level of precision is applied in the paper Eq. (14). Advantage of this STFT iterative procedure is evident comparing to conventional numerical methods, because starting conditions are not needed. Actually, procedure can begin with the value of ψ=1. It has to be underlined that computation complexity is far better than in conventional methods.

Namely, advantage of STFT approaches is conceptual simplicity, absent of boundary conditions and easy numerical implementation.

Let us note that formulae (8) and (11) are very significant for the gradient coefficient $\left(\frac{\partial\psi}{\partial D},\ \frac{\partial\psi}{\partial d}\right)$ genesis, for the theoretical analysis of degree of the nuclear fuel burn-up in nuclides parameters field. Note, in the STFT analysis, that implies all nuclides, appears problem in determining the sumlimts, since the Heaviside's functions are so multidisciplinary.

Finally, we must declare that determining of the degree of the nuclear fuel burn-up, for unlimited number of nuclides (real number of nuclides), will be the subject of our next research.

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REFERENCES

1. V.V. Frolov, *Nuclear-Physical Methods of Fissile Materials Control),* Moskow, Russia: Energoatomizdat, 1989. (В.В. Фролов, *Ядерно-физические методы контроля делящихся веществ*, Москва, Россия: Энергоатомиздат, 1989.)
2. S.M. Perovic and S. Jovanovic, "A New Approach to Evaluation of the Degree of Nuclear Fuel Burn-up," in *Proc. Int. Conf. Signal & Sys.,* vol. 2, Geneva, Switzerland, June 17-19, 1992, pp. 173-180.
3. R. Gregg, A. Worrall, "Effect of Highly Enriched/ Highly Burnt UO_2 Fuels on Fuel Cycle Costs, Radiotoxicity and Nuclear Design Parameters," *Nuclear Technology*, vol. 151, no. 2, pp.126-132, Aug., 2005.
4. J. Secker et al., "Optimum Discharge Burn-up and Cycle Length for PWRs," *Nucl. Technol.*, vol. 151, no.2, pp. 109-111, Aug., 2005.
5. "Experimental Investigation of Burn-up Credit for Safe Transport, Storage, and Disposal of Spent Nuclear Fuel," *NERI 2001 Annual Report*, *NERI 99-200*, US Dept. of Energy, Washington DC, USA, Rep. DOE/NE-0122, pp. 211-212.
6. S.M. Perovich, "Transcendental Method in the Theory of Neutron Slowing Down," *J. Phys. A: Math.Gen.,* vol. 25, no.10, pp. 2969-2689, 1992.
7. S.M. Perovich, "Reply to the Comment on the Transcendental Method," *J. Phys. A: Math.Gen.*, vol. 28, no. 14, pp. 4191-4200, 1995.
8. S.M. Perovich, D.V. Tosic, "Transcendental Method in the Nonlinear Circuits Theory," *Electron. Lett.,* vol. 32, no. 16, pp. 1433-1437, 1996.
9. S.M. Perovich, "Concerning the Analytical Solution of the Disperse Equation in the Linear Transport Theory," *Transp. Theory Stat. Phys.,* vol. 6, no. 6, pp. 705-725, 1997.
10. S.M. Perovich et al., "Some Comments Concerning the Discrete Eigen Value," *Nucl. Sci. Eng.*, vol. 131, no. 3, p. 439, 1999.
11. S.M. Perovich, S.I. Bauk, "The Analytical Analysis of Hopfield Neuron Parameters by the Application of the Special Trans Function Theory," in *Adaptive and Natural Computing Algorithms*, B. Ribeiro, R.F. Albrecht, A. Dobnikar, D.W. Pearson and N.C. Steele, Eds., New York (NY), USA: Springer, pp. 29–32, 2005.
12. S.M. Perovich et al. "On the Analytical Solution of Some Families of Transcendental Equation," *Appl. Math. Lett.,* vol. 20, no. 5, p. 493, 2007.
13. N.S. Singh, et al, "Determination of the Solar Cell Junction Ideality Factor Using Special Trans Function Theory (STFT)," *Sol. Ener. Mat. Sol. Cells*, vol. 93, no. 8, pp. 1423-1426, 2009.
14. S.M. Perovich, S.I. Bauk, "Determination of the Plutonium Temperature Using the Special Trans Functions

Theory," *Nucl. Tech. & Rad. Prot.,* vol. 25, no. 3, pp. 164-170, 2010.

15. S.M. Perovich, S.I. Bauk, "An Inverse Problem of Temperature Estimation for the Combination of the Linear and Non-Linear Resistances," Advan. in Phys., vol. 022110, pp. 1-14, 2011.
16. S.M. Perovich, "The Special Trans Functions Theory," *Research Monograph,* Podgorica, Monte Negro: Univ. of Montenegro, 2004.
17. S.M. Perovich et al., "On the Exact Analytical Solution of Certain Lambert Transcendental Equation," *Math. Probl. Eng.,* vol. 2011, pp. 1-21, 2011.
18. N.S. Singh et al., "A New Method to Determine the Optimum Load of a Real Solar Cell Using Special Trans Function Theory (STFT)," *Int. Jour. Renew. En. Res.,* vol. 3, no. 2, pp. 378-382, 2013.
19. S.M. Perovich, M. Orlandic and M. Calasan, "Concerning Exact Analytical STFT Solution to Some Families of Inverse Problems in Engineering Material Theory," *Appl. Math. Modell.,* vol. 37, no. 7, pp. 5474-5497, Apr. 2013.
20. S. Marguet, *La physique des reacteurs nucleaires*, 1° et 2° éd., Paris, France: Lavoisier, p. 471, 2011, et p. 512, 2013. (S. Marguet, *The Physics of Nuclear Reactors*, 1st and 2nd ed., Paris, France: Lavoisier, p. 471, 2011, and p. 512, 2013.)
21. S.M. Perovich, M. Calasan and R. Toskovic, "On the Exact Analytical Solution of Some Families of Equilibrium Critical Thickness Transcendental Equations," *AIP Advances 4,* vol. 117124, 2014.
22. S.M. Perovich et al., "Concerning a Novel Mathematical Approach to the Solar Cell Junction Ideality Factor Estimation," *Appl. Math. Modell.*, vol. 39, no. 12, pp. 3248-3264, June 2015.
23. S.M. Perovich et al., "Concerning Current STFT Estimations to the RC Diode Circuit," *3rd Mediter. Conf. Embed. Comp.*, Budva, Montenegro, June 15-19, 2014, pp. 193-197.